

## Visualizing Linear Models: An R Bag of Tricks Session 3: Examples & Extensions

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SCS Short Course  
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<https://friendly.github.io/VisMLM-course/>

## Today's topics

- MANOVA examples
  - Distinguishing among psychiatric groups
  - Robust MLMs: down-weighting outliers
- Multivariate regression
  - PA tests & ability
  - Canonical correlation
  - MANCOVA & homogeneity of regression
- Homogeneity of (co)variance
  - Visualizing Box's *M* test

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### Ex: Neuro- & Social-Cognitive measures in psychiatric groups

- A study by Leah Hartman @York examined whether patients classified as 'schizophrenic' or 'schizoaffective' (on DSM-IV) could be distinguished from a normal, control sample on standardized tests in the following domains:
  - **Neuro-Cognitive:** processing speed, attention, verbal learning, visual learning, problem solving
  - **Social-cognitive:** managing emotions, theory of mind, externalizing bias, personalizing bias
- Research questions → MANOVA contrasts
  - Analyze neuro-cog (NC) and social-cog (SC) separately
  - Do the two psychiatric groups differ from the controls?
  - Do the psychiatric groups differ from each other?

**Schizophrenia** symptoms: Hallucinations, disorganized thinking, delusions, ...  
**Schizoaffective** disorder combines symptoms of schizophrenia with mood disorder (bipolar or depression)



See: Friendly & Sigal (2017), Graphical Methods for Multivariate Linear Models in Psychological Research: An R Tutorial *The Quantitative Methods for Psychology*, 13, 20-45, <http://dx.doi.org/10.20982/tamp.13.1.p020>

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# Neuro-cognitive measures

## Data

```
> car::some(NeuroCog)
  Dx Speed Attention Memory Verbal Visual ProbSolv SocialCog Age Sex
14  Schizophrenia 19      9    19    33    24     39     28 44 Female
88  Schizoaffective 27    44    24    33    26     31     36 53 Female
105 Schizoaffective 23    41    42    48    46     29     50 52 Female
114 Schizoaffective 41    53    47    39    30     50     63 32 Female
130 Control        44    25    21    37    32     43     29 43 Female
165 Control        35    35    43    53    57     37     40 30 Male
194 Control        47    40    53    53    40     45     49 55 Female
```

diagnostic group ----- neuro-cognitive measures ----- -- ignored --

## Questions:

- Do the diagnostic groups differ **collectively** on the neuro-cognitive measures?
- How do group differences relate to **research hypotheses**?
- How many **dimensions** (aspects) are reflected in the differences among means?

# Neuro-cognitive measures: Analyses

- Could do a series of univariate ANOVAs
  - `car::Anova(NC.m1m1 <- lm(Speed ~ DX, data=NeuroCog))`
  - `car::Anova(NC.m1m2 <- lm(Attention ~ DX, data=NeuroCog))`
  - `car::Anova(NC.m1m3 <- lm(Memory ~ DX, data=NeuroCog))`
  - ...
- Better: MANOVA test for **all 7 responses together**
  - `Anova(NC.mlm = lm(cbind(Speed, Attention, Memory, ...) ~ Dx, data=SocialCog)`
- Research hypotheses: test contrasts
  - `Dx1: Control - (Schiz + SchizAff)/2 = 0` → `linearHypotheses(NC.mlm, "Dx1")`
  - `Dx2: Schiz - SchAff = 0` → `linearHypotheses(NC.mlm, "Dx2")`
- Dimensions:
  - Canonical analysis → `NC.can <- candisc(NC.mlm); plot(NC.can)`
  - Canonical HE plot → `heplot(NC.can)`
- Tech note: `anova()` in base R vs. `car::Anova`
  - `anova()` uses only Type 1 (sequential) tests, rarely useful; doesn't handle MLM well ❌
  - `car::Anova()` provides Type 2, 3 (partial) tests; give sensible results for MLMs ✅
  - `car::linearHypotheses()` gives univariate and multivariate tests of contrasts ✅

# Neuro-cognitive measures: MANOVA

```
library(heplots); library(car)
data(NeuroCog, package="heplots")

# fit the MANOVA model
NC.mlm <- lm(cbind(Speed, Attention, Memory, Verbal, Visual, ProbSolv) ~ Dx,
             data=NeuroCog)
Anova(NC.mlm)

Type II MANOVA Tests: Pillai test statistic
  Df test stat approx F num Df den Df Pr(>F)
Dx 2 0.2992 6.8902 12 470 1.562e-11 *** ✓
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

So, the groups differ. But how?

What about the research hypotheses?

```
> contrasts(NeuroCog$Dx)
      [,1] [,2]
Schizophrenia -0.5 1
Schizoaffective -0.5 -1
Control 1.0 0
```

Dx1: Control - (Schiz + SchizAff)/2 = 0

# Neuro-cognitive measures: Contrasts

A simple result: Control ≠ (Schizophrenia ≈ Schizoaffective)

```
> print(linearHypothesis(NC.mlm, "Dx1"), SSP=FALSE)

Multivariate Tests:
  Df test stat approx F num Df den Df Pr(>F)
Pillai 1 0.289 15.9 6 234 2.8e-15 *** ✓
Wilks 1 0.711 15.9 6 234 2.8e-15 *** ✓
Hotelling-Lawley 1 0.407 15.9 6 234 2.8e-15 *** ✓
Roy 1 0.407 15.9 6 234 2.8e-15 ***
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

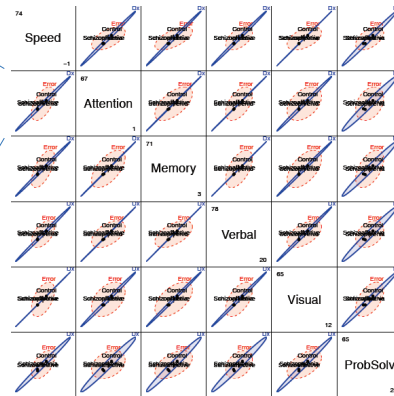
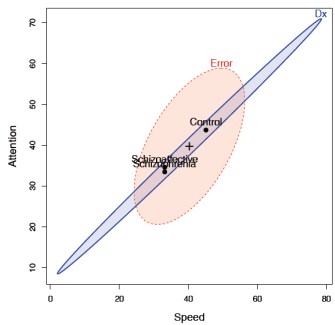
> print(linearHypothesis(NC.mlm, "Dx2"), SSP=FALSE)

Multivariate Tests:
  Df test stat approx F num Df den Df Pr(>F)
Pillai 1 0.006 0.249 6 234 0.96
Wilks 1 0.994 0.249 6 234 0.96
Hotelling-Lawley 1 0.006 0.249 6 234 0.96
Roy 1 0.006 0.249 6 234 0.96 ❌
```

# Visualize me: in data space

# Bivariate view for any 2 responses:  
heplot(NC.mlm, var=1:2, ...)

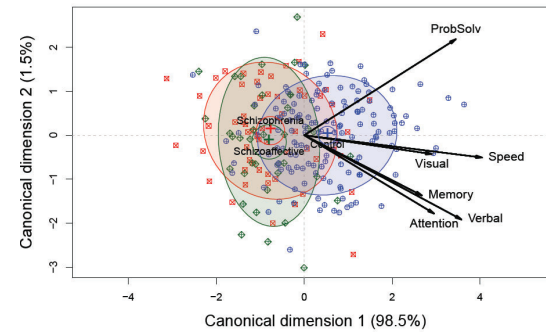
# HE plot matrix: for all responses  
pairs(NC.mlm, ...)



Wow! All neuro-cog measures highly correlated in group means!  
Only 1 dim. of H variation

# Visualize me: in canonical space

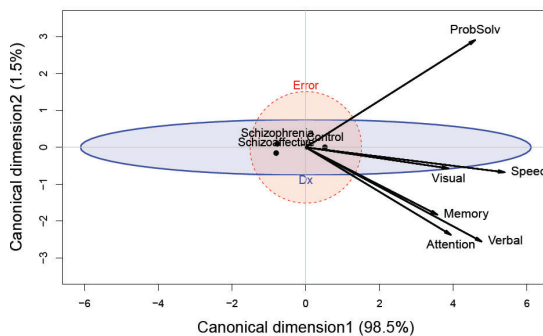
- As with biplot, we can visualize MLM hypothesis variation for *all* responses by projecting  $H$  and  $E$  into low-rank space.
- Canonical projection:  $Y_{n \times p} \mapsto Z_{n \times s} = YE^{-1/2}V$ , where  $V$  = eigenvectors of  $HE^{-1}$ .
- This is the view that maximally discriminates among groups, ie max.  $H$  wrt  $E$ !



Very simple interpretation  
Can1: normal vs. others  
All vars highly + correlated;  
Can2: only 1.5%, NS; but perhaps suggestive (ProbSolv vs. Attention)

# Visualize me: canonical HE plots

- Canonical HE plot is just the HE plot of canonical scores,  $(z_1, z_2)$  in 2D, or  $z_1, z_2, z_3$ , in 3D.
- As in biplot, we add vectors to show relations of the  $y_i$  response variables to the canonical variates.
- variable vectors here are **structure coefficients** = correlations of variables with canonical scores.



The multivariate "juicer"  
Shows just group means, H ellipse & E ellipse  
Variable vectors offer interpretation of Can dimensions.

# Social cognitive measures

- These measures deal with the person's perception and cognitive processing of emotions of others
  - Scales: managing emotions, theory of mind, externalizing bias, personalizing bias
- Questions:
  - Do these differentiate normal from patient groups?
  - Can they distinguish between schizophrenic & schizoffective
  - If so, this could be a major finding.

# Social cognitive measures

## Data

```
> car::some(SocialCog)
      Dx MgeEmotions ToM ExtBias PersBias
24  Schizophrenia      32  18      -2    1.00
58  Schizophrenia      32  17       4    0.92
69  Schizophrenia      52  23       2    0.80
85  Schizoaffective     28  12      -5    1.00
99  Schizoaffective     40  22       1    0.36
111 Schizoaffective     51  23       6    0.91
124  Control            51  24      -5    1.00
155  Control            57  28       1    0.30
168  Control            47  32       0    0.77
216  Control            64  31       3    0.69
```

diagnostic group ----- social-cognitive measures -----

## Analyses

Standard MANOVA test:

```
Anova(SC.mlm = lm(cbind(...) ~ DX, data=SocialCog)
```

Test contrasts:

```
linearHypotheses(SC.mlm, "Dx1")
linearHypotheses(SC.mlm, "Dx2")
```

```
> contrasts(SocialCog$Dx)
      [,1] [,2]
Schizophrenia -0.5  1
Schizoaffective -0.5 -1
Control      1.0  0
```

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# Social cognitive measures

```
> data(SocialCog, package="heplots")
> SC.mlm <- lm(cbind(MgeEmotions,ToM, ExtBias, PersBias) ~ Dx,
              data=SocialCog)
> Anova(SC.mlm)
```

```
Type II MANOVA Tests: Pillai test statistic
Df test stat approx F num Df den Df Pr(>F)
Dx 2      0.212      3.97      8    268 0.00018 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```



Test contrasts: Dx1 = Normal vs. Patient; Dx2 = Schizo vs. Schizoaffective

```
> print(linearHypothesis(SC.mlm, "Dx1"), SSP=FALSE)
Multivariate Tests:
```

```
Pillai      Df test stat approx F num Df den Df Pr(>F)
1          1  0.1355      5.212      4    133 0.000624 ***
```



```
> print(linearHypothesis(SC.mlm, "Dx2"), SSP=FALSE)
Multivariate Tests:
```

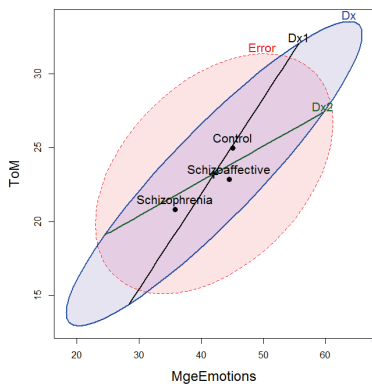
```
Pillai      Df test stat approx F num Df den Df Pr(>F)
1          1  0.0697      2.493      4    133 0.0461 *
```



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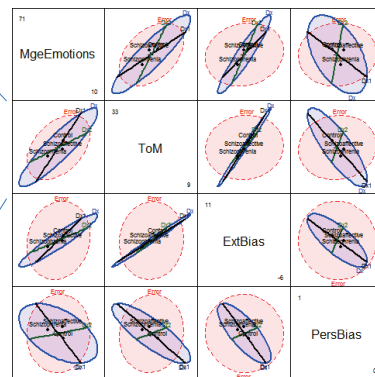
# Visualize me: data space

```
heplot(SC.mlm,
        hypotheses=list("Dx1", "Dx2"),...)
```



The groups are ordered Schizo < ScAffective < Control on these measures

```
pairs(SC.mlm,
       hypotheses=list("Dx1", "Dx2"),...)
```



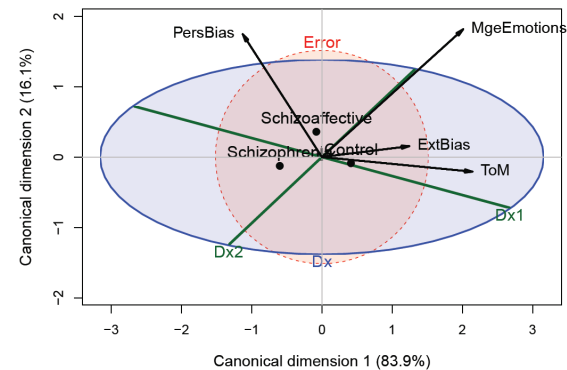
One of these vars is not like the others!

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# Visualize me: canonical space

Contrasts:

Dx1 : Control vs. patients. Controls > patients on MgeEmotions, ExtBias, ToM  
Dx2 : Schizo vs. schizoaffective.



Can1: group order  
Can2: Schizoaffective vs. others  
Dx1 suggests ≈ spacing Schizo < ScAff < Control

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## Model checking & remedies

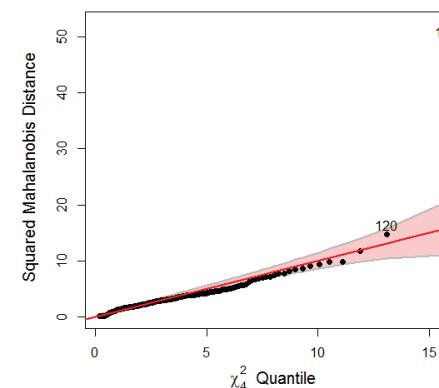
- The MLM assumes residuals are multivariate normal
  - Squared Mahalanobis distances
 
$$D_M^2(\mathbf{y}_i) = (\mathbf{y}_i - \bar{\mathbf{y}})^T \mathbf{S}^{-1} (\mathbf{y}_i - \bar{\mathbf{y}}) \sim \chi_p^2$$
 with  $p$  d.f.
  - a quantile – quantile plot of ordered  $D_M^2(\mathbf{y}_i)$  vs. quantiles of  $\chi_p^2$  should plot as **straight line**
  - Outliers are readily apparent
  - plots: `heplots::cqplot()`
- Influence plots
  - `mvinfluence::influence()` calculates multivariate analogs of influence measures

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## Social cog: cqplot

`heplots::cqplot()` creates a chi-square QQ plot from a MLM

```
cqplot(SC.mlm, id.n=2)
```



One observation appears as an extreme outlier.

This was a case w/ ExtBias = -33, but valid range = (-10, +10)

Refitting w/o case 15:  
Overall & DX1 tests still OK 😊  
Dx2 test:  $p=0.074$ , now NS ☹️

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## Social cog: Influence

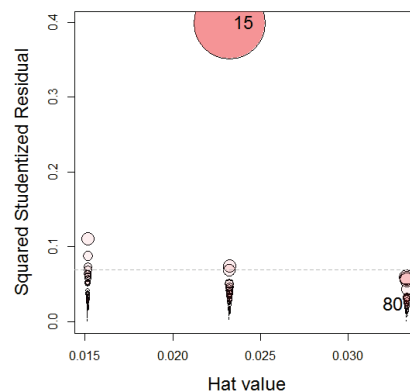
`mvinfluence::influencePlot()` creates a multivariate analog of an influence plot

```
>library(mvinfluence)
>influencePlot(SC.mlm, ...)
```

	H	Q	CookD	L	R
15	0.0233	0.3985	0.4202	0.0238	0.4080
80	0.0333	0.0233	0.0352	0.0345	0.0241

Case 15 stands out as hugely influential

The 3 columns of circles correspond to the 3 groups



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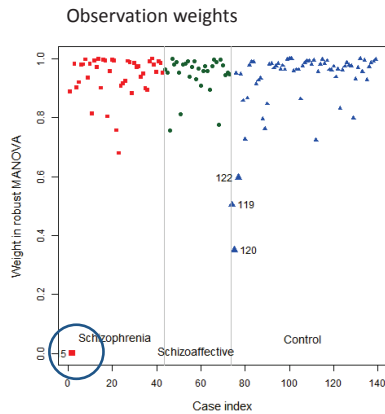
## Robust MLMs

- Robust methods for univariate LMs are now well-developed and implemented
  - proper SEs, CIs and hypothesis tests
- Analogous methods for multivariate LMs are a current hot research topic
- The `heplots` package now provides `robmlm()` for the fully general MLM (MANOVA, MMReg)
  - Uses simple M-estimator via IRLS
  - Weights: calculated from Mahalanobis  $D^2$ : a robust covariance estimator and weight function,  $\psi(D^2)$
$$D^2 = (\mathbf{Y} - \bar{\mathbf{Y}})^T \mathbf{S}_{\text{robust}}^{-1} (\mathbf{Y} - \bar{\mathbf{Y}}) \sim \chi_p^2$$
  - Downside: SEs,  $p$ -values only approximate

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## Robust MLMs

```
SC.rlm <- robmlm(cbind( MgeEmotions, ToM, ExtBias, PersBias) ~ Dx,
data=SocialCog)
```



Approx test of Dx2 in robust model

```
> print(linearHypothesis(SC.rlm, "Dx2"), SSP=FALSE)
Multivariate Tests:
      Df test stat approx F num Df den Df Pr(>F)
Pillai 1  0.069  2.44      4  132  0.05 *
Wilks  1  0.931  2.44      4  132  0.05 *
...
```

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## Robust MLMs: Pottery data

- Chemical composition of ancient pottery found at four sites in Great Britain

- Sites: AshleyRails, Caldicot, IsleThorns, Llanedyrn
- Measures: Al Fe Mg Ca Na

- Analyses:

- standard MANOVA `lm(cbind(Al, Fe, Mg, ...) ~ Site, data=Pottery)`
- robust MANOVA `robmlm(cbind(Al, Fe, Mg, ...) ~ Site, data=Pottery)`

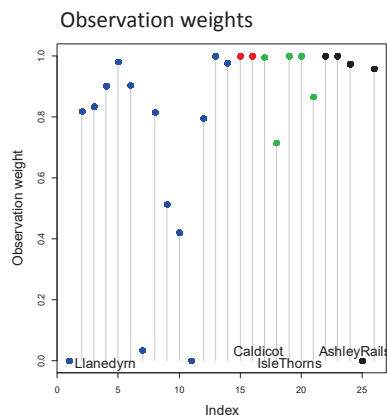
- Data:

```
> car::some(Pottery)
      Site Al Fe Mg Ca Na
2  Llanedyrn 13.8 7.08 3.43 0.12 0.17
14 Llanedyrn 12.5 6.44 3.94 0.22 0.23
17 IsleThorns 18.3 1.28 0.67 0.03 0.03
18 IsleThorns 15.8 2.39 0.63 0.01 0.04
21 IsleThorns 20.8 1.51 0.72 0.07 0.10
22 AshleyRails 17.7 1.12 0.56 0.06 0.06
23 AshleyRails 18.3 1.14 0.67 0.06 0.05
```

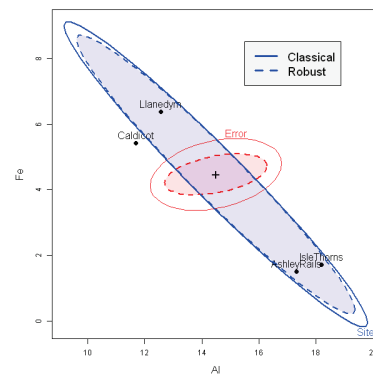
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## Robust MLMs: Pottery data

```
> pottery.mod <- lm(cbind(Al,Fe,Mg,Ca,Na)~Site, data=Pottery)
> pottery.rmod <- robmlm(cbind(Al,Fe,Mg,Ca,Na)~Site, data=Pottery)
```



residual E ellipse shrinks a lot  
overlaid HE plots



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## MMRA example: PA tasks & ability

- Rohwer data from Timm (1975)
- How well do paired associate (PA) tasks predict performance on measures of aptitude & achievement in kindergarten children?

- Samples: 69 children in two groups (schools): 'Lo' | 'Hi' SES
- Outcomes (Y):
  - Scholastic aptitude test (SAT)
  - Peabody picture vocabulary test (PPVT)
  - Raven progressive matrices (Raven)
- Predictors (X): Scores (0—40) on PA tasks where the stimuli were:
  - named (n), still (s), named-still (ns), named-action (na), sentence-still (ss)

```
> data("Rohwer", package="heplots")
> car::some(Rohwer, n=5)
      group SES SAT PPVT Raven n s ns na ss
8  1 Lo 8 68 8 0 0 10 19 14
9  1 Lo 49 74 11 0 0 7 16 13
17 1 Lo 19 66 13 7 12 21 35 27
52 2 Hi 38 66 14 0 0 3 16 11
66 2 Hi 8 55 16 4 7 19 20 13
```

Having a group factor makes the analysis more complicated (MANCOVA)

Start with analysis of the Hi SES group

```
> Rohwer2 <- subset(Rohwer, subset=SES=="HI")
```

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# Why not univariate models?

```
rohwer.mod1 <- lm(SAT ~ n + s + ns + na + ss, data = Rohwer2)
rohwer.mod2 <- lm(PPVT ~ n + s + ns + na + ss, data = Rohwer2)
rohwer.mod3 <- lm(Raven ~ n + s + ns + na + ss, data = Rohwer2)
```

```
library(stargazer)
stargazer(rohwer.mod1, rohwer.mod2, rohwer.mod3, type="text", ...)
```

Univariate regression models for Rohwer data

	Dependent variable:		
	SAT	PPVT	Raven
n	3.26*	0.07	0.06
s	3.00	0.37	0.49**
ns	-5.86***	-0.37	-0.16
na	5.67***	1.52*	0.12
ss	-0.62	0.41	-0.12
R2	0.56	0.35	0.31
F Statistic (df = 5; 26)	6.54***	2.85*	2.32

Note: \*p<0.05; \*\*p<0.01; \*\*\*p<0.001

Results are disappointing

- Only model for SAT highly signif.
- Only a few coeffs. signif. ≠0

# MANOVA tests

```
> rohwer.mlm <- lm(cbind(SAT, PPVT, Raven) ~ n + s + ns + na + ss, data=Rohwer2)
> Anova(rohwer.mlm)
```

Type II MANOVA Tests: Pillai test statistic

	Df	test	stat	approx	F	num	Df	den	Df	Pr(>F)
n	1	0.202	2.02	3	24	0.1376				
s	1	0.310	3.59	3	24	0.0284 *				
ns	1	0.358	4.46	3	24	0.0126 *				
na	1	0.465	6.96	3	24	0.0016 **				
ss	1	0.089	0.78	3	24	0.5173				

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Much better!

- Multivariate tests:
- pool evidence for all Ys
  - take correlations of Ys into account

Can also test overall hypothesis,  $H_0: \mathbf{B} = \mathbf{0}$  (all coeffs = 0)

```
> print(linearHypothesis(rohwer.mlm,
+ c("n", "s", "ns", "na", "ss")), SSP=FALSE)
```

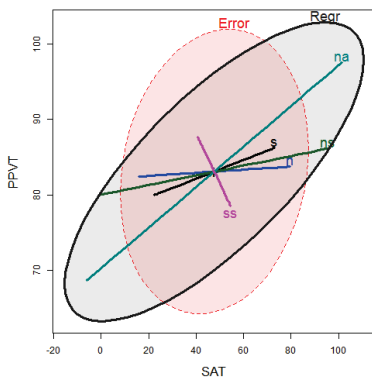
Multivariate Tests:

	Df	test	stat	approx	F	num	Df	den	Df	Pr(>F)
Pillai	5	1.0386	2.753	15	78.00	0.001912 **				
Wilks	5	0.2431	2.974	15	66.65	0.001154 **				
Hotelling-Lawley	5	2.0615	3.115	15	68.00	0.000697 ***				
Roy	5	1.4654	7.620	5	26.00	0.000160 ***				

Strongly reject  $H_0$  by all criteria

# Visualize me!

```
cols <- c("red", "blue", "black", "darkgreen", "darkcyan", "magenta", "gray20")
hyp <- list("Regr" = c("n", "s", "ns", "na", "ss")) # Test of B = 0
heplot(rohwer.mlm,
       hypotheses = hyp,
       fill=TRUE, fill.alpha=0.1, col=cols, lwd=c(1,3))
```



Each predictor gives a 1 df test ->  $H_i$  ellipse is a line  
 $\mathbf{E}$  here is a 3D ellipsoid ( $\text{rank}(\mathbf{E}) = \min(p,q)$ )

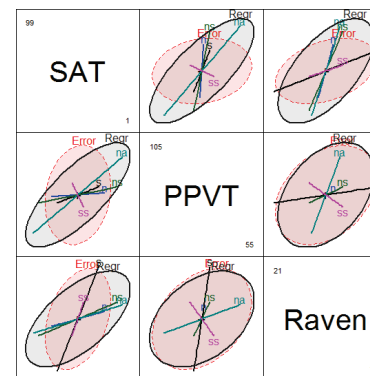
Interpretation:

- Any  $H_i$  ellipse that protrudes outside  $\mathbf{E}$  ellipse is significant by Roy's test
- Length of each  $H_i$  line  $\approx$  strength of association
- Orientation of each  $H_i$  line shows relation of  $X_i$  to the two  $Y_j$  responses shown.

# pairs.mlm() plot

View all pairwise HE plots

```
pairs(rohwer.mlm, hypotheses=hyp,
      col=cols, fill=TRUE, fill.alpha=0.1, ...)
```



Can now understand more subtle aspects

SAT is best predicted overall, but relation with PA tests varies  
 The na & ns tasks are strongest for SAT

Raven is weakly predicted

## Canonical correlations

For quantitative (X, Y) data, canonical correlation analysis is an alternative to MMRA  
It finds the weighted sums of the Y variables most highly correlated with the Xs

```
> X <- Rohwer2[, 6:10] # X variables for High SES students
> Y <- Rohwer2[, 3:5] # Y variables for High SES students
> (cc <- cancel(X, Y, set.names=c("PA", "Ability")))
```

Canonical correlation analysis of:  
5 PA variables: n, s, ns, na, ss  
with 3 Ability variables: SAT, PPVT, Raven

CanR	CanRSQ	Eigen	percent	cum	scree
1	0.7710	0.5944	1.4654	71.080	71.08 *****
2	0.5465	0.2987	0.4259	20.659	91.74 *****
3	0.3815	0.1455	0.1703	8.261	100.00 ***

Test of H0: The canonical correlations in the current row and all that follow are zero

CanR	LR	test stat	approx F	numDF	denDF	Pr(> F)
1	0.77096	0.24307	2.9738	15	66.655	0.001154 **
2	0.54652	0.59926	1.8237	8	50.000	0.094538 .
3	0.38147	0.85448	1.4759	3	26.000	0.244178

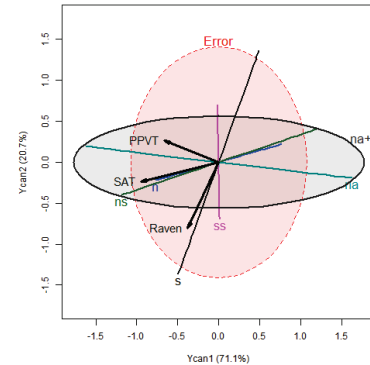
Two dimensions acct for 91.7% of (X,Y) association

Only Can1 is significant

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## Visualize CCA in HE plot

```
cols <- c("red", "blue", "black", "darkgreen", "darkcyan",
          "magenta", "gray20")
heplot(cc, hypotheses=list("na+ns"=c("na", "ns")),
       fill = TRUE, fill.alpha=0.1, col=cols,
       label.pos = c(3, rep(1,5), .1),
       cex=1.4, var.cex=1.25, var.lwd=3, var.col="black")
```



Residuals are uncorrelated in canonical space

H ellipses for X terms same as in ordinary HE plots – outside E ellipse iff signif. by Roy's test

Variable vectors for Ys: correlations with canonical variables Ycan1, Ycan2

- SAT & PPVT: mainly Ycan1
- Raven: more aligned with Ycan2

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## MANCOVA & homogeneity of regression

- With a group variable (SES) can test differences in means (intercepts)
  - `rohwer.mod <- lm(cbind(SAT, PPVT, Raven) ~ SES + n + s + ns + na + ss, data=Rohwer)`
  - This assumes that slopes (B) are the same for both groups (homogeneity of regression)
- Can test for equal slopes by adding interactions of SES with Xs
  - `rohwer.mod1 <- lm(cbind(SAT, PPVT, Raven) ~ SES * (n + s + ns + na + ss))`
- Or, fit separate models for each group

```
rohwer.ses1 <- lm(cbind(SAT, PPVT, Raven) ~ n + s + ns + na + ss,
                 data = Rohwer, subset = SES == "Hi")
rohwer.ses2 <- lm(cbind(SAT, PPVT, Raven) ~ n + s + ns + na + ss,
                 data = Rohwer, subset = SES == "Lo")
```

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## MANCOVA

Fit the MANCOVA model & test hypotheses

```
> rohwer.mod <- lm(cbind(SAT, PPVT, Raven) ~ SES + n + s + ns + na + ss,
                  data=Rohwer)
> Anova(rohwer.mod)
```

Type II MANOVA Tests: Pillai test statistic

	Df	test stat	approx F	num Df	den Df	Pr(>F)	
SES	1	0.379	12.18	3	60	2.5e-06 ***	SES: diff in means
n	1	0.040	0.84	3	60	0.4773	
s	1	0.093	2.04	3	60	0.1173	
ns	1	0.193	4.78	3	60	0.0047 **	
na	1	0.231	6.02	3	60	0.0012 **	
ss	1	0.050	1.05	3	60	0.3770	

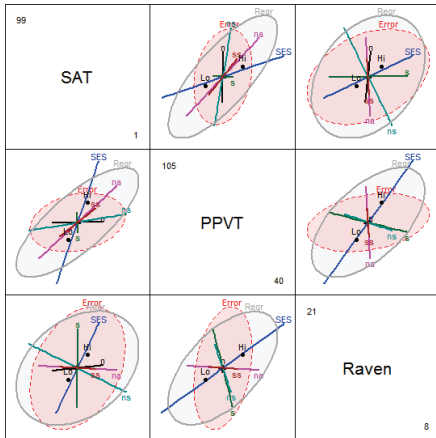
---  
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

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## Visualize effects

```
pairs(rohwer.mod,
      hypotheses=list("Regr" = c("n", "s", "ns", "na", "ss")),
      fill=TRUE, fill.alpha=0.1)
```



The SES effect is positive for all Y variables  
Hi SES group > Lo SES group

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## Fit model with interactions

Fit heterogeneous regression model with SES interactions

```
> rohwer.mod1 <- lm(cbind(SAT, PPVT, Raven) ~ SES * (n + s + ns + na + ss),
+                   data=Rohwer)
> Anova(rohwer.mod1)
```

Type II MANOVA Tests: Pillai test statistic							
	Df	test stat	approx F	num Df	den Df	Pr(>F)	
SES	1	0.391	11.78	3	55	4.5e-06	***
n	1	0.079	1.57	3	55	0.20638	.
s	1	0.125	2.62	3	55	0.05952	.
ns	1	0.254	6.25	3	55	0.00100	***
na	1	0.307	8.11	3	55	0.00015	***
ss	1	0.060	1.17	3	55	0.32813	.
SES:n	1	0.072	1.43	3	55	0.24417	.
SES:s	1	0.099	2.02	3	55	0.12117	.
SES:ns	1	0.118	2.44	3	55	0.07383	.
SES:na	1	0.148	3.18	3	55	0.03081	*
SES:ss	1	0.057	1.12	3	55	0.35094	.

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

OK, as expected

Hmm ???

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## Test interactions

Can test all interactions **simultaneously** with linearHypothesis()  
Do I need any interaction terms?

I use a 'grep' trick here to find the names of coefficients like 'SES:' containing a ':'

```
> coefs <- rownames(coef(rohwer.mod1)) # store coefficient names in a vector
> print(linearHypothesis(rohwer.mod1, # only test for interaction effects
+                        coefs[grep(":", coefs)]), SSP=FALSE)
```

Multivariate Tests:							
	Df	test stat	approx F	num Df	den Df	Pr(>F)	
Pillai	5	0.4179	1.845	15	171.0	0.03209	*
wilks	5	0.6236	1.894	15	152.2	0.02769	*
Hotelling-Lawley	5	0.5387	1.927	15	161.0	0.02396	*
Roy	5	0.3846	4.385	5	57.0	0.00191	**

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

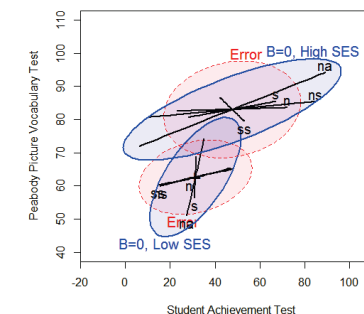
Evidence shows that **some** slopes differ for Hi/Lo SES

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## Fit separate models

Fitting a model for each group allows all slopes to differ  
Also allows within-group covariances to differ

```
rohwer.ses1 <- lm(cbind(SAT, PPVT, Raven) ~ n + s + ns + na + ss,
                  data = Rohwer, subset = SES == "Hi")
rohwer.ses2 <- lm(cbind(SAT, PPVT, Raven) ~ n + s + ns + na + ss,
                  data = Rohwer, subset = SES == "Lo")
```



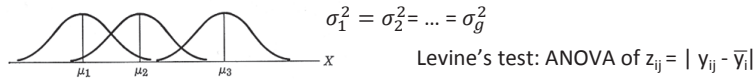
For SAT & PPVT:

- means higher for Hi SES
- within-group covariance larger for Hi SES
- slopes of predictors smaller for Hi SES  
→ SAT more important for this group.

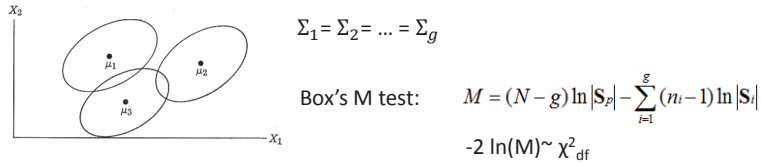
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# Homogeneity of (co)variances

- ANOVA assumes equality of residual variances



- MANOVA: assumes equality of covariance matrices

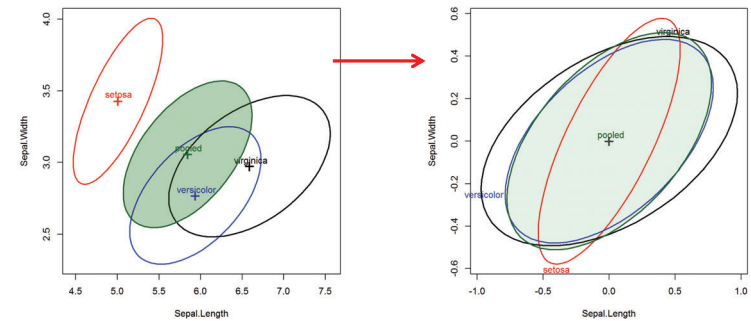


See: <http://www.datavis.ca/papers/EqCov-TAS.pdf>

# Visualizing covariance matrices

Visualize covariance ellipses in data space  
Center to see pure differences in size & shape

```
covEllipses(iris[,1:2], iris$Species, ...)
covEllipses(iris[,1:2], iris$Species, center=TRUE, ...)
```

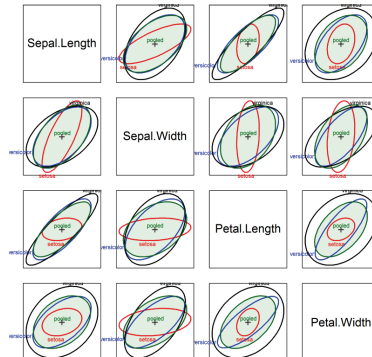


# View all pairs

```
covEllipses(iris[,1:4], iris$Species,
center=TRUE, variables=1:4, ...)
```

In all cases, **setosa** stands out as different from the others

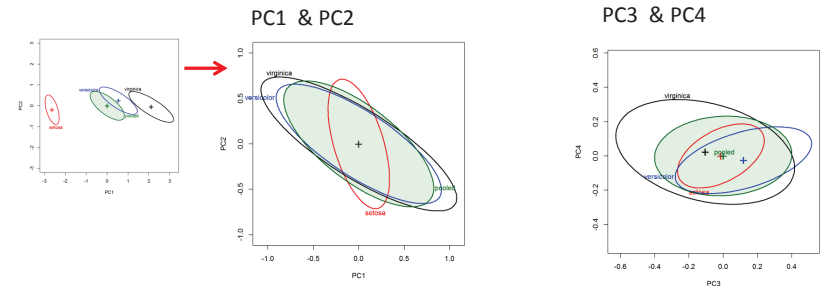
- sometimes correlation differs
- sometimes smaller variance(s)



# Visualize in PCA space

PCA projects the data into an orthogonal space accounting for maximum variance  
Covariance ellipses show the differences among groups in this space

Surprisingly, the **small** dimensions contribute largely to Box's M test.



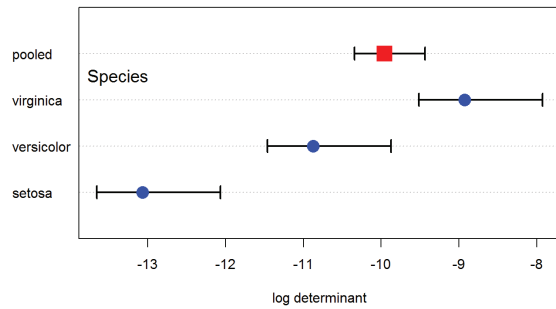
```
iris.pca <- prcomp(iris[,1:4])
covEllipses(iris.pca$x, iris$Species, ...)
covEllipses(iris.pca$x, iris$Species, center=TRUE, ...)
```

```
covEllipses(iris.pca$x, iris$Species, center=TRUE,
variables=3:4, ...)
```

## Visualizing Box's $M$ test

Box's test is based on a comparison of the  $\log |S_i|$  relative to  $\log |S_p|$ : **plot them!**

```
iris.boxm <- boxM(iris[, 1:4], iris[, "Species"])
plot(iris.boxm, glabel="Species")
```



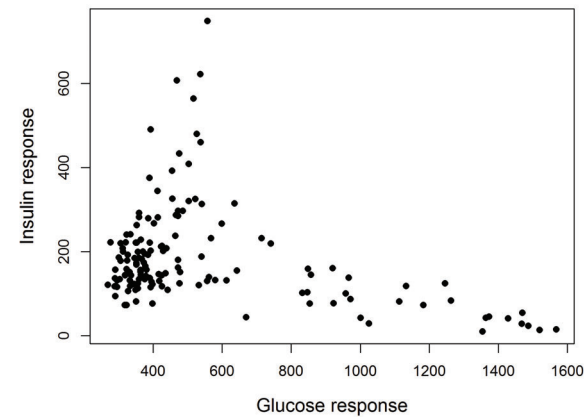
CIs based on an asymptotic CLT  $\approx$  distribution of  $\ln|S|$  (Cai, Liang, and Zhou 2016) (Thx: Augustine Wong)

Unsolved: Bootstrap CI

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## Diabetes data: 2D mystery

Reaven & Miller (1968) found a peculiar "horse shoe" result in analysis of data on the relationship of blood glucose levels and production of insulin in patients with varying degrees of hyperglycemia



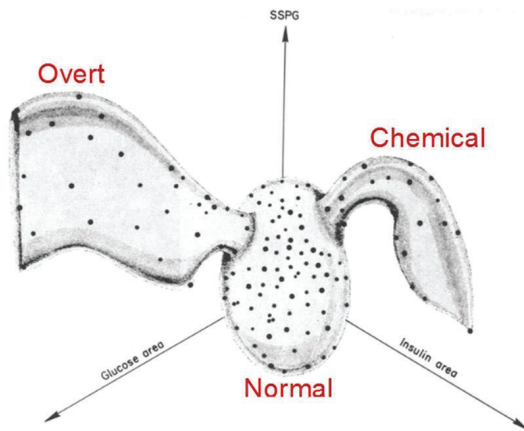
In a 2D plot this was a medical mystery.

What could be the explanation?

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## Diabetes data: 3D clarity

Using the first 3D computer graphics system (PRIM-9) they rotated the data in 3-space until a hypothesis was suggested.



Artist's view of the data suggests there were actually three groups in the data.

Two categories of Type 2 diabetes:

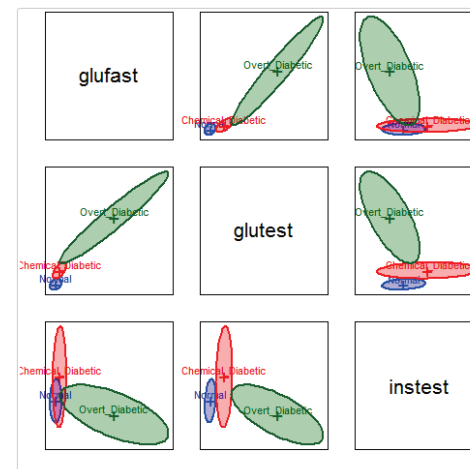
- Overt (advanced)
- Chemical (latent)

But, these were NOT stages in a progression!

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## Diabetes data: Ellipses

```
covEllipses(Diabetes[,2:5], Diabetes$group, fill=TRUE, pooled=FALSE,
col=c("blue", "red", "darkgreen"), variables=1:3)
```



The pairwise data ellipses show visual summaries of the data

The differences in correlation and variances are dramatic here

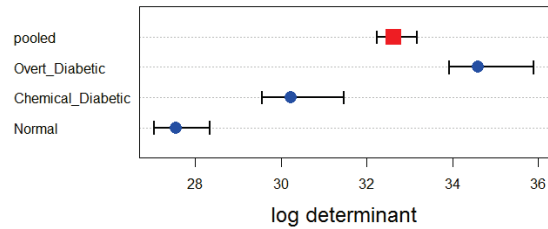
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## Box's $M$ test

```
diab.boxm <- boxM(Diabetes[,2:5], Diabetes$group)
diab.boxm
##
## Box's M-test for Homogeneity of Covariance Matrices
##
## data: Diabetes[, 2:5]
## Chi-Sq (approx.) = 383, df = 20, p-value <2e-16

plot(diab.boxm)
```

All pairs differ!



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## Summary

- MANOVA tests of MLMs are easily visualized in HE plots
  - Contrasts among groups can be easily shown
  - Canonical plots show data in 2D/3D space of max. group differences
  - Robust methods can help guard against outliers
- MMRA models
  - Visualize effects of quant. predictors as lines in data space
  - Test & visualize any linear hypothesis
  - Canonical correlations: visualize in 2D/3D of max. (X, Y) correlations
- Homogeneity of covariances
  - Visualize within-group  $S_i$  and pooled  $S_p$  by data ellipses
  - Visualize Box's  $M$  test by simple dot plot of  $|S_p|$  and  $|S_i|$

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